THE SCOTS COLLEGE

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 1999

4 UNIT MATHEMATICS

TIME ALLOWED: THREE HOURS

INSTRUCTIONS TO CANDIDATES:

- ALL QUESTIONS ARE TO BE ATTEMPTED.
- ALL QUESTIONS ARE OF EQUAL VALUE.
- ALL NECESSARY WORKING SHOULD BE SHOWN FOR EACH QUESTION.
- A STANDARD TABLE OF INTEGRALS IS PROVIDED.
- APPROVED CALCULATORS MAY BE USED.
- EACH SECTION [A, B, C AND D] IS TO BE DONE IN A SEPARATE BOOKLET, CLEARLY MARKED SECTION A, SECTION B, ETC.

NOTE: This is a **TRIAL PAPER** only and does not necessarily reflect the content or format of the final HSC examination paper for this subject.

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SECTION A

QUESTION 1

A function $f(x)$ is defined by $f(x) = \frac{\log_e x}{x}$ for $x > 0$.

(a) Prove that the graph of $f(x)$ has a relative maximum turning point at $x = e$ and a point of inflexion at $x = e^{\frac{3}{2}}$.

(b) Discuss the behaviour of $y = f(x)$ in the neighbourhood of $x = 0$ and for large values of x.

(c) Hence draw a clear sketch of $f(x)$ indicating on it all these features.

(d) Draw separate sketches of:

(i) $y = \left| \frac{\log_e x}{x} \right|$ **(ii)** $y = \frac{x}{\log_e x}$

(Note: There is no need to find any further derivatives for this part)

(e) What is the range of the function: $y = \frac{x}{\log_e x}$?

QUESTION 2

(a) Express $z = 2 + 2\sqrt{3}\,i$ in modulus-argument form, and hence express each of the following in the form $a + ib$.

(i) $\frac{1}{z}$ **(ii)** z^5

(b) Find the square roots of 5 - 12i.

(c) Sketch, on separate Argand Diagrams, the locus of z described by each of the following conditions:

(i) $|z-i| = |z-3|$

(ii) $0 < \arg(z-i) < \frac{2\pi}{3}$

(iii) $\arg\left(\frac{z-i}{z-1}\right) = \frac{\pi}{2}$

(iv) $z\bar{z} = z + \bar{z}$

(d) Let OABC be a square on an Argand Diagram where O is the origin. The points A and C represent the complex numbers z and iz respectively.

Find the complex number represented by B.

SECTION B

QUESTION 3

(a) (i) Show that $\frac{x^3}{x^2+2} = x - \frac{2x}{x^2+2}$

(ii) Find the exact value of $\int_0^2 \frac{x^3}{x^2+2}\,dx$

(b) By a suitable change of variable, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin x}}\,dx$

(c) (i) Show that $\tan^{-1} 3 - \tan^{-1} \frac{1}{2} = \frac{\pi}{4}$

(ii) Evaluate $\int_1^6 \frac{dx}{4+x^2}$ in terms of π.

(d) By using partial fractions find the value of $\int_2^5 \frac{2(x+1)}{(x-1)(2x-1)}\,dx$

(e) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\;dx$, show $I_n = \frac{n-1}{n} I_{n-2}$, and hence find the value of I_6.

QUESTION 4

(a) Reduce $\left(x^2+2x\right)^2-9$ to irreducible factors over the real number field.

(b) $P(x)$ is a real polynomial of least degree such that $P(i)=P(\frac{1}{2})=0$

Express $P(x)$ in general polynomial form.

(c) If $x^3+2x-1=0$ has roots α,β,γ find the value of $\alpha^3+\beta^3+\gamma^3$.

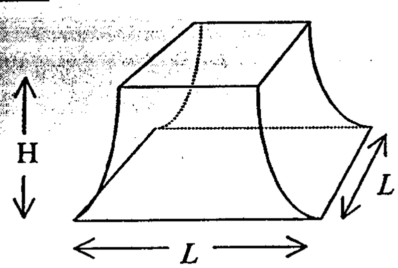
(d) By considering the stationary values of $f(x)=x^3-3px^2+4q$, where p and q are positive real constants, show that the equation $f(x)=0$ has three real, distinct roots if $p^3>q$.

(e) If $G(x)$ is an odd function, then $G(x)=-G(-x)$. Use this definition to show that $\log_e\left(x+\sqrt{x^2+1}\right)$ is an odd function.

SECTION C

QUESTION 5

(a)

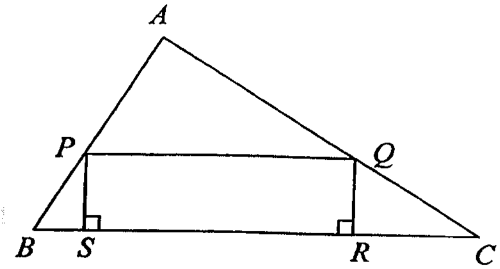


A stone monument of height H has the shape of a flat topped square "pyramid" with curved sides as shown in the figure.

The cross section at height h metres is a square with sides parallel to the sides of the base and of length $l(h)=\dfrac{L}{\sqrt{(h+1)}}$ where L is the side length of the square base in centimetres.

Find the volume of the monument given that $L=H=30$cm; giving your answer to the nearest cubic centimetre.

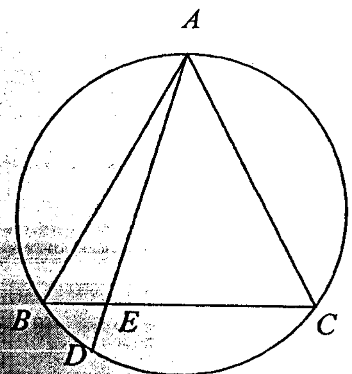
(b)



In the figure PQ is parallel to BC, and $PQRS$ is a rectangle. Prove that the maximum area of $PQRS$ is half the area of the triangle ABC.

QUESTION 6

(a)



An isosceles triangle ABC is inscribed in a circle.

$AB = AC$ and the chord AD intersects BC at E.

(i) Copy the diagram into your answer booklet.

(ii) By means of a suitable construction and using Pythagoras' Theorem, or otherwise, prove $AB^2 - AE^2 = BE.EC$

(b) Two circles with centres O and P and radii r and s (where $r < s$) respectively touch externally at T. ABC and ADE are common tangents to the circles, with B, C, D, E lying on the circles.

(i) Draw a neat diagram to show this information.

(ii) Show that A, O, T and P are collinear.

(iii) Show that $AO = \dfrac{r(r+s)}{s-r}$

SECTION D

QUESTION 7

(a) Given that $z = \cos\theta + i\sin\theta$, use de Moivre's Theorem to show that:

$$z^n + z^{-n} = 2\cos n\theta$$

Hence, or otherwise, solve the equation:

$$2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$$

(b) Given $P(x) = x^4 - 2x^3 + 2x - 1 = 0$ has a root of multiplicity 3, find the factors of $P(x)$.

(c) If one root of the equation $x^3 - px^2 + qx - r = 0$ is equal to the product of the other two, show that $(q+r)^2 = r(p+1)^2$.

QUESTION 8

(a) If $u_1 = 5, u_2 = 13$ and $u_n = 5u_{n-1} - 6u_{n-2}$ for $n \geq 3$, show by induction that $u_n = 2^n + 3^n$ for n ...

(b) **(i)** If $a > 0$, $b > 0$, show that $a + b \geq 2\sqrt{ab}$.

(ii) Hence show that:

(α) If $a > 0, b > 0, c > 0$ then $(a+b)(b+c)(c+a) \geq 8abc$

(β) If $a > 0, b > 0, c > 0, d > 0$ then $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{d} + \dfrac{d}{a} \geq 4$

(c) **(i)** Show that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$ for all real θ.

(ii) Hence find in surd form, the value of $\cot\dfrac{\pi}{8} + \cot\dfrac{\pi}{12}$ and show that

$$\operatorname{cosec}\frac{2\pi}{15} + \operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} = 0.$$

END OF PAPER

QUESTION 1.

(a)

$$y = \frac{\ln x}{x}$$

$$y' = \frac{x\left(\frac{1}{x}\right) - 1.\ln x}{x^2}$$

$$= \frac{1-\ln x}{x^2}$$

$$= 0 \quad \text{for stat. pts}$$

$$\therefore \quad 1 - \ln x = 0$$

$$\ln x = 1 \quad \Rightarrow \quad x = e$$

$$y'' = \frac{x^2\left(-\frac{1}{x}\right) - 2x(1-\ln x)}{x^4}$$

$$= \frac{-x - 2x + 2x\ln x}{x^4}$$

$$= \frac{-3x + 2x\ln x}{x^4}$$

$$= \frac{2\ln x - 3}{x^3}$$

$$\text{at } x = e, \quad y'' = \frac{2\ln e - 3}{e^3}$$

$$= \frac{2-3}{e^3} \quad \text{, since } \ln e = 1$$

$$< 0$$

$\therefore$ a relative maximum occurs when $x = e$

(4)

a point of inflexion occurs for $y'' = 0$

$$\therefore \quad \frac{2\ln x - 3}{x^3} = 0$$

$$\Rightarrow \quad \ln x = \frac{3}{2}$$

$$\therefore \quad x = e^{\frac{3}{2}}$$

Change of sign

$\therefore$ a point of inflexion occurs for $x = e^{\frac{3}{2}}$

(b) Let $x = \frac{p}{q} \to 0^+ \Rightarrow p <<< q$, $q \to \infty$.

$$\therefore y = \frac{\ln \frac{p}{q}}{\frac{p}{q}}$$

$$= q\left(\frac{\ln p - \ln q}{p}\right)$$

$\therefore y \to -\infty$ since $\ln p - \ln q < 0$, q large compared to $\frac{\ln p - \ln q}{p}$

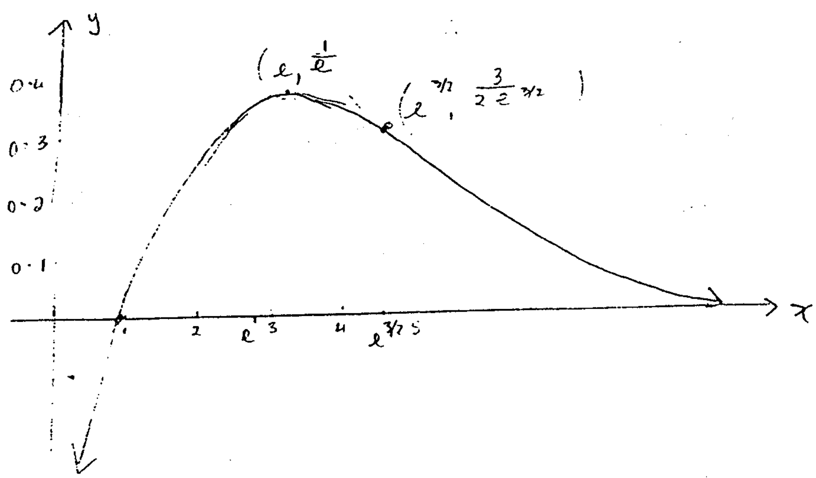
(scientifically rel... ln x being dominated by x

Let $x = \frac{p}{q} \to \infty \Rightarrow p >>> q$, $p \to \infty$

$$y = q\left(\frac{\ln p - \ln q}{p}\right)$$

$\to 0^+$ since $\ln p - \ln q > 0$, p large compared to $q(\ln p - \ln q)$

(c)



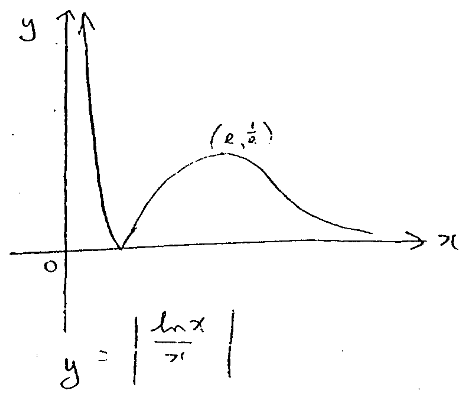
at $x = e$, $y = \frac{\ln e}{e} = \frac{1}{e} \approx 0.36$

at $x = e^{3/2}$, $y = \frac{\ln e^{3/2}}{e^{3/2}} = \frac{\frac{3}{2}}{e^{3/2}}$

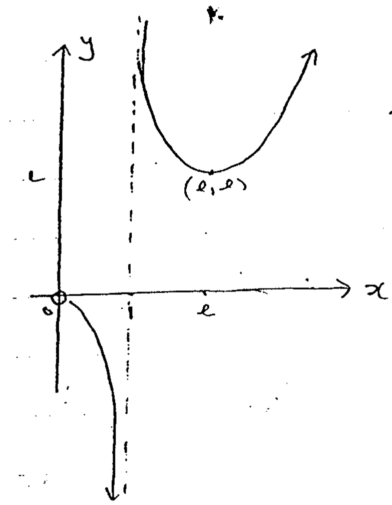
when $y = 0$, $x = 1$

(d)

(i)



(ii)



(e) From (d)(ii), the range of $y = \frac{x}{\ln x}$ can be seen to be {all real y, excluding $0 < y < e$}

QUESTION 2

(a) $z = 2 + 2\sqrt{3}\,i$

$= r(\cos\theta + i\sin\theta)$ in mod-arg form.

where $r = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{16} = 4.$

and $\tan\theta = \frac{2\sqrt{3}}{2} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

$\therefore\ z = (4\cos\pi/3 + i\sin\pi/3)$

(i) $\frac{1}{z} = z^{-1} = 4^{-1}\left[\cos(-\pi/3) + i\sin(-\pi/3)\right]$, using De Moivre's Th.

$\frac{1}{4}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{8} - \frac{\sqrt{3}}{8}\,i$

(ii) $z^5 = 4^5(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$

$= 1024\left[\cos(-\pi/3) + i\sin(-\pi/3)\right]$

$= 1024\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= 512 - 512\sqrt{3}\,i$

(b) Let $z = x + iy = \sqrt{5 - 12i}$

$\therefore (x + iy)^2 = 5 - 12i$

$x^2 - y^2 + 2xyi = 5 - 12i$

$\Rightarrow x^2 - y^2 = 5$ and $xy = -6.$

$y = -\frac{6}{x}$

$\therefore x^2 - \frac{36}{x^2} = 5$

$x^4 - 5x^2 - 36 = 0.$

$(x^2 - 9)(x^2 + 4) = 0$

$\therefore x^2 - 9 = 0$ and $x^2 + 4 = 0 \Rightarrow$ no real solutions

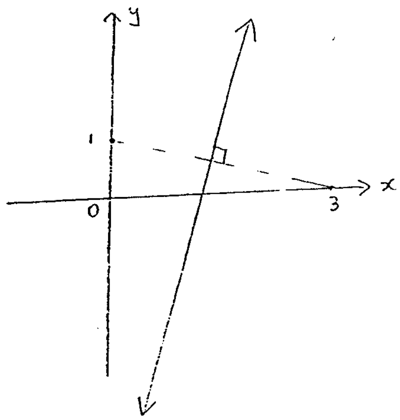
$\therefore x = \pm 3$

$y = \mp 2$

$\therefore \sqrt{5 - 12i} = \pm(3 - 2i)$

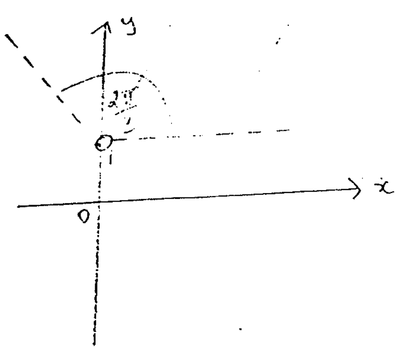
(c) (i) $|z - i| = |z - 3|$

$\Rightarrow |x + (y-1)i| = |(x-3) + yi|$



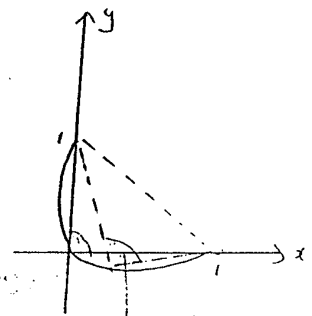
The required locus is the perp. bisector of the interval joining $(0,1)$ and $(3,0)$ and has eq. $3x - y - 4 = 0$

(ii)



2ea

(iii)



$\arg\left(\frac{z-i}{z-1}\right) = \pi/2$

$$\arg\left(\frac{z-i}{z-1}\right) = \pi/2$$

$$\Rightarrow \quad \arg(z-i) - \arg(z-1) = \pi/2$$

required locus is a semicircle with diameter joining $(0,1)$ and $(1,0)$

(iv) $z\bar{z} = z + \bar{z}$

Let $z = x + iy$

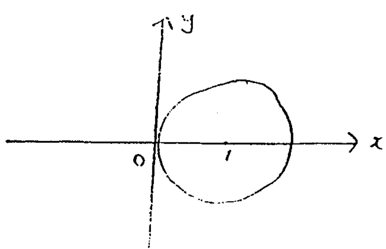
$\therefore (x+iy)(x-iy) = (x+iy) + (x-iy)$

$x^2 + y^2 = 2x$

$\therefore x^2 - 2x + y^2 = 0$

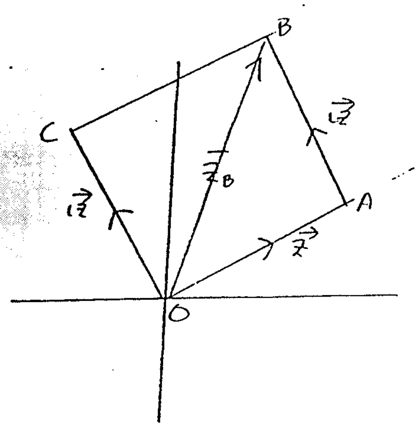
$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$



The required locus is a circle centre $(1, 0)$, radius 1

(d)



Let the vectors $\vec{z}$, $i\vec{z}$, $\vec{z}_B$ represent the points A, C, B.

Since C is represented by iz, this is equivalent to OA being rotated clockwise through 90°

$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\therefore \quad \vec{OA} + \vec{OC} = \vec{OB} \quad \text{since } \vec{OC} = \vec{AB}$$

$$\vec{z} + i\vec{z} = \vec{z}_B$$

ie B represents the complex number $z + iz$

QUESTION 3.

a) (i) $\frac{x^3}{x^2+2} = \frac{x^3+2x-2x}{x^2+2}$

$= \frac{x(x^2+2)-2x}{x^2+2}$

$= x - \frac{2x}{x^2+2}$

(ii) $\int_0^2 \frac{x^3}{x^2+2}\,dx = \int_0^2 \left(x - \frac{2x}{x^2+2}\right)dx$

$= \left[\frac{x^2}{2} - \ln(x^2+2)\right]_0^2$

$= \left(\frac{4}{2} - \ln 6\right) - (0 - \ln 2)$

$= 2 - \ln 6 + \ln 2$ or $2 - (\ln 6 - \ln 2)$

$= 2 + \ln 2 - \ln 6$ $= 2 - \ln \frac{6}{2}$

$= 2 + \ln\left(\frac{1}{3}\right)$ $= 2 - \ln 3$

(b) Let $1 + \sin x = u$ at $x = 0$, $u = 1$

$\cos x\,dx = du$ at $x = \pi/2$, $u = 2$

$\int_0^{\pi/2} \frac{\cos x\,dx}{1+\sin x} = \int_1^2 \frac{du}{u^{1/2}}$

$= \int_1^2 u^{-1/2}\,du$

$= \left[\frac{u^{1/2}}{1/2}\right]_1^2$

$= 2\left[\sqrt{2} - 1\right]$

(c) $\displaystyle\int_1^6 \frac{dx}{4+x^2} = \int_1^6 \frac{dx}{2^2+x^2}$

$$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_1^6$$

$$= \frac{1}{2}\left(\tan^{-1}\frac{6}{2} - \tan^{-1}\frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\tan^{-1}3 - \tan^{-1}\frac{1}{2}\right)$$ ← Some [illegible] hint here?

Let $\tan^{-1}3 = \alpha$, $\tan^{-1}\frac{1}{2} = \beta$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$= \frac{3-\frac{1}{2}}{1+3\left(\frac{1}{2}\right)}$$

$$= \frac{5/2}{5/2}$$

$$= 1$$

$\therefore \alpha - \beta = \pi/4$

$$\therefore \int_1^6 \frac{dx}{4+x^2} = \frac{1}{2}\left(\tan^{-1}3 - \tan^{-1}\tfrac{1}{2}\right)$$

$$= \frac{1}{2}(\alpha-\beta)$$

$$= \frac{1}{2}\cdot\frac{\pi}{4}$$

$$= \pi/8$$

(a) Let $\frac{2(x+1)}{(x-1)(2x-1)} = \frac{a}{x-1} + \frac{b}{2x-1}$

$$\therefore\ 2(x+1) = a(2x-1) + b(x-1)$$

Let $x = \frac{1}{2}$: $2\left(\frac{3}{2}\right) = 0 + b\left(-\frac{1}{2}\right) \Rightarrow b = -6$

Let $x = 1$: $2(2) = a(1) + 0 \Rightarrow a = 4$

$$\therefore\ \frac{2x+1}{(x-1)(2x-1)} = \frac{4}{x-1} - \frac{6}{2x-1}$$

$$\int_2^5 \frac{2x+1}{(x-1)(2x-1)}\,dx = \int_2^5 \left(\frac{4}{x-1} - \frac{6}{2x-1}\right) dx$$

$$= \Big[4\ln(x-1) - 3\ln(2x-1)\Big]_2^5$$

$$= (4\ln 4 - 3\ln 9) - (4\ln 1 - 3\ln 3)$$

$$= 4\ln 4 - 3(\ln 9 - \ln 3)$$

$$= 4\ln 4 - 3\ln \frac{9}{3}$$

$$= 4\ln 4 - 3\ln 3$$

$$= \ln\left(\frac{4^4}{3^3}\right)$$

$$\ln\left(\frac{256}{27}\right)$$

e. $I_n = \int_0^{\pi/2} \sin^n x \, dx$

$= \int_0^{\pi/2} \sin x \sin^{n-1} x \, dx$

$= \left[-\cos x \sin^{n-1} x \right]_0^{\pi/2} - \int_0^{\pi/2} (-\cos x)(n-1)\sin^{n-2} x \cos x \, dx$

$= 0 + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x$

$= (n-1) \int_0^{\pi/2} \sin^{n-2} x \, (1 - \sin^2 x) \, dx$

$= (n-1) \left[\int_0^{\pi/2} \sin^{n-2} x \, dx - \int_0^{\pi/2} \sin^2 x \, dx \right]$

$= (n-1) \left[I_{n-2} - I_n \right]$

$\therefore I_n + (n-1) I_n = (n-1) I_{n-2}$

$n I_n = (n-1) I_{n-2}$

$\therefore I_n = \frac{n-1}{n} I_{n-2}$

$I_6 = \frac{5}{6} I_4$

$= \frac{5}{6} \left(\frac{3}{4} I_2 \right)$

$= \frac{5}{8} \left(\frac{1}{2} I_0 \right)$

$= \frac{5}{16} I_0$

$= \frac{5}{16} \cdot \frac{\pi}{2}$

$= \frac{5\pi}{32}$

$I_0 = \int_0^{\pi/2} (\sin x)^0 \, dx$

$= \int_0^{\pi/2} 1 \, dx$

$= \frac{\pi}{2}$

QUESTION 4

(a) $(x^2+2x)^2 - 9 = (x^2+2x-3)(x^2+2x+3)$

$= (x+3)(x-1)(x^2+2x+3)$, since x^2+2x+3 is reducible only over the complex field

(b) $P(i) = P(\frac{1}{2}) = 0 \Rightarrow (x-i)$ and $(x-\frac{1}{2})$ i.e $(2x-1)$ are factors!

Since $P(x)$ is real and of least degree, the conjugate of $(x-i)$ i.e $(x+i)$ is a factor.

$\therefore P(x) = (x-i)(x+i)(2x-1)$

$= (x^2+1)(2x-1)$

$= 2x^3 - x^2 + 2x - 1$

(c). Since $x^3+2x-1=0$, has roots α, β, γ then

$\alpha^3 + 2\alpha - 1 = 0$

$\beta^3 + 2\beta - 1 = 0$

$\gamma^3 + 2\gamma - 1 = 0$

$\therefore \alpha^3+\beta^3+\gamma^3 + 2(\alpha+\beta+\gamma) - 3 = 0.$

But $\alpha+\beta+\gamma = -\frac{b}{a} = 0.$

$\therefore \alpha^3+\beta^3+\gamma^3 + 0 - 3 = 0$

$\therefore \alpha^3+\beta^3+\gamma^3 = 3.$

(d). $f(x) = x^3 - 3px^2 + 4q$

$f'(x) = 3x^2 - 6px$

$= 3x(x-2p)$

$= 0$ for stat pts.

$\therefore x = 0, 2p$

$\therefore$ stationary points exist for $x = 0, 2p$.

$f''(x) = 6x - 6p$

at $x = 0$, $f''(0) = -6p < 0$ (since $p>0$) $\Rightarrow$ max at $x=0$

at $x = 2p$, $f''(2p) = 6p > 0$ (since $p>0$) $\Rightarrow$ min at $x = 2p$

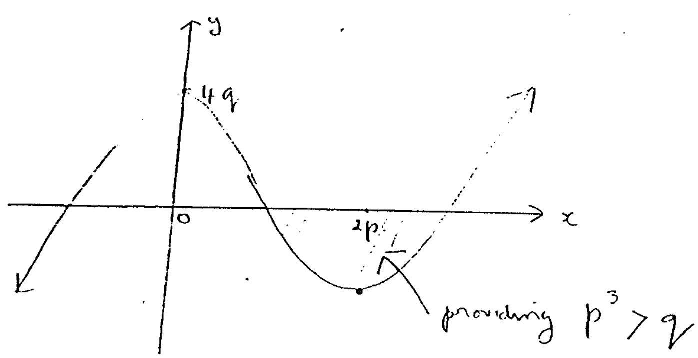
at $x = 0$; $f(0) = 4q$ $\therefore$ the stat pt is above the x axis

at $x = 2p$, $f(2p) = 8p^3 - 12p^3 + 4q = 4(q-p^3)$

For three distinct real roots to occur, the minimum at $x = 2p$ must be below the x axis ie. $f(2p) < 0 \Rightarrow q - p^3 < 0$

ie $p^3 > q$

a possible graph is shown below.



) Let $G(x) = \log_e\left(x + \sqrt{x^2+1}\right)$

$$-G(-x) = -1 \log_e\left(-x + \sqrt{(-x)^2+1}\right)$$

$$= \log_e\left(-x + \sqrt{x^2+1}\right)^{-1}$$

$$= \log_e\left(\frac{1}{-x+\sqrt{x^2+1}}\right)$$

$$= \log_e\left(\frac{1}{(\sqrt{x^2+1} - x)} \times \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)$$

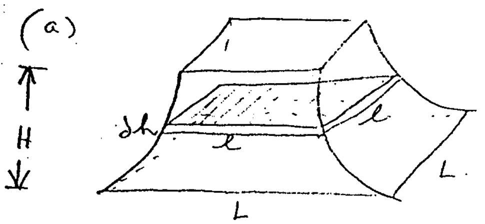
$$= \log_e\left(\frac{x+\sqrt{x^2+1}}{x^2+1-x^2}\right)$$

$$= \log_e\left(x + \sqrt{x^2+1}\right)$$

$$= G(x)$$

QUESTION 5

(a)



Consider a horizontal slice of side length $l(h)$ and thickness δh taken parallel to the base
Let δV be its volume

$$\delta V = \left[l(h)\right]^2 \delta h$$
$$= \left(\frac{L}{\sqrt{h+1}}\right)^2 \delta h$$

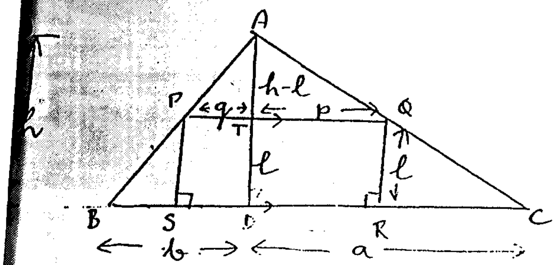
$$\therefore V = \lim_{\delta h \to 0} \sum_{0}^{H} \frac{L^2}{h+1} \delta h$$

$$= \int_0^H \frac{L^2}{h+1} \, dh$$

$$= L^2 \left[\ln(h+1)\right]_0^H$$

$$= L^2 \ln(H+1)$$

If $L = H = 30$, $V = 30^2 \ln 31$
$= 3091 \text{ cm}^3$



Drop a perp from A to D. Let AD = h. Let the remaining lengths be as marked on the diagram.

Since the sides of the rectangles are parallel to the base and the vertical height, the sets of triangles are similar

In Δ's ADC, ATQ

$$\frac{p}{a} = \frac{h-l}{h} \Rightarrow p = \frac{a}{h}(h-l)$$

In Δ's, ADB, ATP $\quad \frac{q}{b} = \frac{h-l}{h} \rightarrow q = \frac{b}{h}(h-l)$

$$A_{PQRS} = (p+q)\,l = \left[\frac{a}{h}(h-l) + \frac{b}{h}(h-l)\right] l$$

$$= \left(\frac{a+b}{h}\right)(h-l)\,l$$

$$= \left(\frac{a+b}{h}\right)(hl - l^2)$$

Or use ratio of areas = (ratio sides)^2

(combine ΔPBS and QRC)

$$\frac{dA_{PQRS}}{dl} = \left(\frac{a+b}{h}\right)(h-2l)$$

$= 0$ for st values

$\therefore h = 2l$

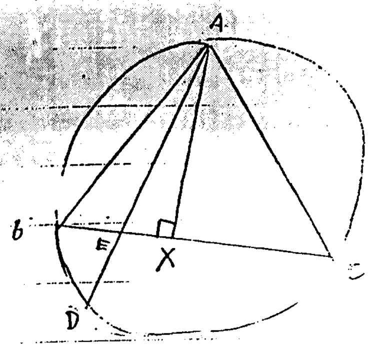
$$\frac{d^2A}{dl^2} = \left(\frac{a+b}{h}\right)(-2) < 0 \Rightarrow \text{max. area when } h = 2l$$

$$\therefore p = \frac{a}{2l}(2l-l) = \frac{a}{2} \quad \text{and} \quad q = \frac{b}{2}$$

$$\frac{A_{PQRS}}{A_{\Delta ABC}} = \frac{(p+q)\,l}{\frac{1}{2}(a+b)\,h} = \frac{\left(\frac{a}{2}+\frac{b}{2}\right)l}{\left(\frac{a}{2}+\frac{b}{2}\right)2l} = \frac{1}{2}$$

$\therefore$ max area is ½ of the triangle

QUESTION 6



Let $AX \perp BC$

By Pythagoras' Th.

$$AB^2 = AX^2 + BX^2$$
$$= AE^2 - EX^2 + BX^2$$
$$\therefore AB^2 - AE^2 = -EX^2 + BX^2$$
$$= -EX^2 + (BE + EX)^2$$
$$= -EX^2 + BE^2 + 2BE.EX + EX^2$$
$$= BE(BE + 2EX)$$

Since ABC is isos the perp bisector AX bisects the base

$\therefore BX = XC$

$BE + EX = EC - EX$

$\therefore BE + 2EX = EC$

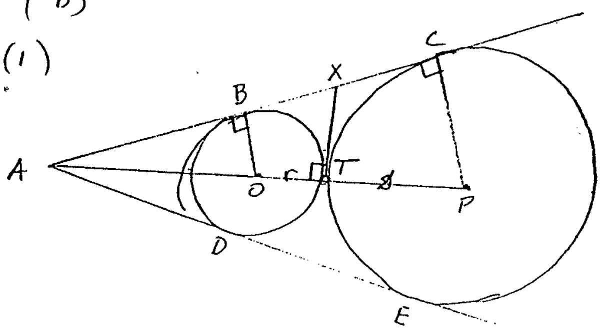
From above

$\therefore AB^2 - AE^2 = BE(BE + 2EX)$

$= BE.EC$

(b)

(i)



(ii) Since the circles touch at one point (T) only, then a tangent at T exists for both circles. Let this tangent be TX

For the circle centre O, $\angle OTX = 90^\circ$ (tangent $\perp$ radius)
For the circle centre P $\angle PTX = 90^\circ$ (tangent $\perp$ radius)

$\therefore \angle OTP = \angle OTX + \angle PTX = 180^\circ$

$\therefore$ O, T, P are collinear.

Since OBXT is a cyclic quadrilateral ($\angle OBX$, $\angle OTX$ both 90°)

$\therefore \angle BOT + \angle BXT = 180^\circ$ (opp $\angle$'s in a cyc quad supplementary)

Since $\triangle$s BAO, XAT are similar ($\angle ABO = \angle XTA = 90^\circ$, $\angle BAO$ common)

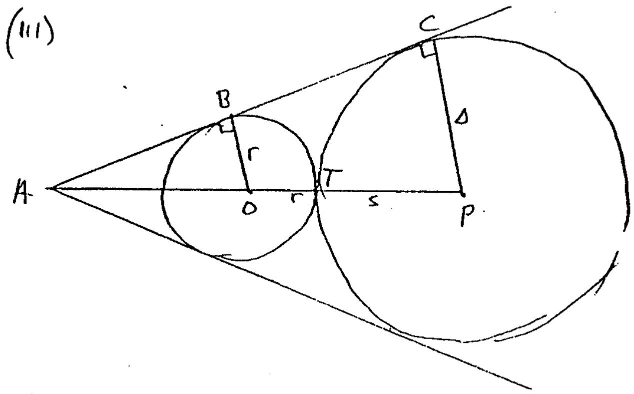
then $\angle BOA = \angle AXT$ ($= \angle BXT$)

$\therefore \angle BOT + \angle BOA = 180^\circ$ (from above)

$\therefore$ A, O, T are collinear

and $\therefore$ A, O, T, P are collinear

(iii)



For the Δ's ABO, ACP

$\angle ABO = \angle ACP$ ($= 90^\circ$ tangent/radius)

$\angle BAO = \angle CAP$ (common)

$\therefore \ \Delta ABO \ ||| \ \Delta ACP$. (angle sum of Δ, corresp eq. angles)

$$\therefore \frac{AO}{AP} = \frac{OB}{CP}$$

$$\frac{AO}{AO+OP} = \frac{r}{s}$$

$$\frac{AO}{AO+(r+s)} = \frac{r}{s}$$

$$\therefore (AO)s = (AO)r + r(r+s)$$

$$\therefore AO(s-r) = r(r+s)$$

$$\text{i.e. } AO = \frac{r(r+s)}{s-r}$$

QUESTION 7

(a) $z = \cos\theta + i\sin\theta$

$$z^n = \cos n\theta + i\sin n\theta$$

$$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos n\theta - i\sin n\theta$$

$$\therefore z^n + z^{-n} = 2\cos n\theta$$

$$2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$$

Dividing by z^2

$$2z^2 + 3z + 5 + \frac{3}{z} + \frac{2}{z^2} = 0$$

$$2\left(z^2 + \frac{1}{z^2}\right) + 3\left(z + \frac{1}{z}\right) + 5 = 0$$

$$2\left(z^2 + z^{-2}\right) + 3\left(z + z^{-1}\right) + 5 = 0$$

$$2\left(2\cos 2\theta\right) + 3(2\cos\theta) + 5 = 0$$

$$4\cos 2\theta + 6\cos\theta + 5 = 0$$

$$4\left(2\cos^2\theta - 1\right) + 6\cos\theta + 5 = 0$$

$$8\cos^2\theta + 6\cos\theta + 1 = 0$$

$$(4\cos\theta + 1)(2\cos\theta + 1) = 0$$

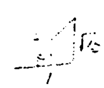
$$\therefore \cos\theta = -\tfrac{1}{2}, -\tfrac{1}{4}$$

For $\cos\theta = -\frac{1}{2}$

$\sin\theta = \pm\frac{\sqrt{3}}{2}$

For $\cos\theta = -\frac{1}{4}$

$\sin\theta = \pm\frac{\sqrt{15}}{4}$



$\therefore$ Solutions are $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \ -\frac{1}{4} \pm \frac{\sqrt{15}}{4}$

(b). $P(x) = x^4 - 2x^3 + 2x - 1 = (x-\alpha)^3 Q_1(x)$

$P'(x) = 4x^3 - 6x^2 + 2 = (x-\alpha)^2 Q_2(x)$

$P''(x) = 12x^2 - 12x = 12x(x-1) = (x-\alpha)\, Q_3(x)$

$\therefore \alpha = 0, 1$

The zero must satisfy $P(x)$, $P'(x)$ $\therefore \alpha = 1$ $\left\{ \begin{array}{l} P(0) \neq 0 \\ P(1) = P'(1) = 0 \end{array} \right\}$

$\therefore x^4 - 2x^3 + 2x - 1 = (x-1)^3(x+a)$

By inspection, $a = 1$

D'

$\therefore P(x) = (x-1)^3(x+1)$

(c). Let the roots be α, β, γ and $\alpha = \beta\gamma$

Sum of roots $\alpha + \beta + \gamma = p \Rightarrow \beta\gamma + \beta + \gamma = p$ — (1)

Sum of pairs $\alpha\beta + \beta\gamma + \alpha\gamma = q \Rightarrow \beta^2\gamma + \beta\gamma + \beta\gamma^2 = q$ — (2)

Product $\alpha\beta\gamma = r \Rightarrow \beta\gamma\beta\gamma = r$ — (3)

From (2) $\beta\gamma(\beta+\gamma) + \beta\gamma = q \rightarrow \beta\gamma(\beta+\gamma+1) = q$

D from (3) $(\beta\gamma)^2 = r \rightarrow \beta\gamma = \sqrt{r}$ 5

$\sqrt{r}\,(\beta+\gamma+1) = q$

From (1) $\sqrt{r} + \beta + \gamma = p \rightarrow \beta + \gamma = p - \sqrt{r}$

$\therefore \sqrt{r}\,(p - \sqrt{r} + 1) = q$

$\sqrt{r}\,(p+1) - r = q$

$\sqrt{r}\,(p+1) = q + r$

$r(p+1)^2 = (q+r)^2$

QUESTION 8

assume $u_n = 2^n + 3^n$ is true for $n \geq 1$

For $n = 1$, $u_1 = 2 + 3 = 5$ $\therefore$ true

For $n = 2$, $u_2 = 2^2 + 3^2 = 13$ $\therefore$ true

given
assume $u_k = 5u_{k-1} - 6u_{k-2}$ is true for all $n = k$

$$\begin{aligned} \therefore u_{k+1} &= 5u_k - 6u_{k-1} \\ &= 5(2^k + 3^k) - 6(2^{k-1} + 3^{k-1}) \\ &= 5.2^k + 5.3^k - 3.2^k - 2.3^k \\ &= 2.2^k + 3.3^k \\ &= 2^{k+1} + 3^{k+1} \end{aligned}$$ which is true for all $n = k+1$.

Since u_n is true for $n = 1$, it is true for $n = 1+1 = 2$.
Since u_n is true for $n = 2$, it is true for $n = 2+1 = 3$,
and so on $\therefore$ u_n is true for all $n \geq 1$

(b) (i)
$$\begin{aligned} (a-b)^2 &= a^2 + b^2 - 2ab \geq 0 \\ \therefore a^2 + b^2 &\geq 2ab \\ \therefore a^2 + 2ab + b^2 &\geq 4ab \\ \therefore (a+b)^2 &\geq 4ab \\ a + b &\geq 2\sqrt{ab} \end{aligned}$$ ($\because a > 0, b > 0$)

(ii) (α)
$$\begin{aligned} a + b &\geq 2\sqrt{ab} \\ b + c &\geq 2\sqrt{bc} \\ c + a &\geq 2\sqrt{ca} \end{aligned}$$

$$\begin{aligned} \therefore (a+b)(b+c)(c+a) &\geq 8\sqrt{ab}\sqrt{bc}\sqrt{ca} \\ &\geq 8\sqrt{a^2b^2c^2} \\ &\geq 8abc \end{aligned}$$

(b) (ii) (β)

$$\left(\frac{a}{b}+\frac{b}{c}\right)+\left(\frac{c}{d}+\frac{d}{a}\right) \geqslant 2\sqrt{\frac{a}{b}\cdot\frac{b}{c}}+2\sqrt{\frac{c}{d}\cdot\frac{d}{a}}$$
$$= 2\sqrt{\frac{a}{c}}+2\sqrt{\frac{c}{a}}$$
$$= 2\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)$$
$$\geqslant 2\cdot 2\sqrt{\sqrt{\frac{a}{c}}\cdot\sqrt{\frac{c}{a}}}$$
$$= 4\sqrt{1}$$

i.e. $\left(\frac{a}{b}+\frac{b}{c}\right)+\left(\frac{c}{d}+\frac{d}{a}\right) \geqslant 4$

c (i) (LHS =) $\operatorname{cosec} 2\theta + \cot 2\theta$

$$= \frac{1}{\sin 2\theta}+\frac{\cos 2\theta}{\sin 2\theta}$$
$$= \frac{1+\cos 2\theta}{\sin 2\theta}$$
$$= \frac{2\cos^2\theta}{2\cos\theta\sin\theta} \qquad \text{using } \cos 2\theta = 2\cos^2\theta - 1$$
$$= \frac{\cos\theta}{\sin\theta}$$
$$= \cot\theta$$

(ii) $\cot\frac{\pi}{8} = \operatorname{cosec}\frac{\pi}{4}+\cot\pi/4 = \sqrt{2}+1$

$\cot\frac{\pi}{12} = \operatorname{cosec}\pi/6+\cot\pi/6 = 2+\sqrt{3}$

$\therefore \cot\pi/8+\cot\pi/12 = 3+\sqrt{2}+\sqrt{3}$

(iii) $\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta$

$\operatorname{cosec}\frac{2\pi}{15} = \cot\pi/15 - \cot\frac{2\pi}{15}$

$\operatorname{cosec}\frac{4\pi}{15} = \cos\frac{2\pi}{15} - \cot\frac{4\pi}{15}$

$\operatorname{cosec}\frac{8\pi}{15} = \cot\frac{4\pi}{15} - \cot\frac{8\pi}{15}$

$\operatorname{cosec}\frac{16\pi}{15} = \cot\frac{8\pi}{15} - \cot\frac{16\pi}{15}$

$\therefore \operatorname{cosec}\frac{2\pi}{15}+\operatorname{cosec}\frac{4\pi}{15}+\operatorname{cosec}\frac{8\pi}{15}+\operatorname{cosec}\frac{16\pi}{15} = \cot\pi/15 - \cot\frac{16\pi}{15}$

$$= \cot\frac{\pi}{15} - \cot\left(\pi + \frac{\pi}{15}\right)$$

$$= \cot\frac{\pi}{15} - \cot\frac{\pi}{15}$$

$$= 0$$